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Abstract

Let $n \geq 3$. We prove that the quaternion group of order 8 is realised as a subgroup of the sphere braid group $B_n(\mathbb{S}^2)$ if and only if n is even. If n is divisible by 4 then the commutator subgroup of $B_n(\mathbb{S}^2)$ contains such a subgroup. Further, for all $n \geq 3$, $B_n(\mathbb{S}^2)$ contains a subgroup isomorphic to the dicyclic group of order $4n$.

The braid groups B_n of the plane were introduced by E. Artin in 1925 [A1, A2], and were generalised by Fox to braid groups of arbitrary topological spaces using the notion of configuration space [FoN]. Van Buskirk showed that the braid groups of a compact connected surface M possess torsion elements if and only if M is the sphere $\mathbb{S}^2$ or the real projective plane $\mathbb{R}P^2$ [VB]. Let us recall briefly some of the properties of the braid groups of the sphere [FVB, GVB, VB].

If $\mathbb{D}^2 \subseteq \mathbb{S}^2$ is a topological disc, there is a group homomorphism $\iota\colon B_n \to B_n(\mathbb{S}^2)$ induced by the inclusion. If $\beta \in B_n$ then we shall denote its image $\iota(\beta)$ simply by β. Then $B_n(\mathbb{S}^2)$ is generated by $\sigma_1, \ldots, \sigma_{n-1}$ which are subject to the following relations:

$$\begin{gathered}
\sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| \geq 2 \text{ and } 1 \leq i,j \leq n-1\\
\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ for all } 1 \leq i \leq n-2, \text{ and}\\
\sigma_1 \cdots \sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2} \cdots \sigma_1 = 1.
\end{gathered}$$

Consequently, $B_n(\mathbb{S}^2)$ is a quotient of B_n. The first three sphere braid groups are finite: $B_1(\mathbb{S}^2)$ is trivial, $B_2(\mathbb{S}^2)$ is cyclic of order 2, and $B_3(\mathbb{S}^2)$ is a ZS-metacyclic group (a group

2000 AMS Subject Classification: 20F36 (primary) whose Sylow subgroups, commutator subgroup and commutator quotient group are all cyclic) of order 12. The Abelianisation of $B_n(\mathbb{S}^2)$ is isomorphic to the cyclic group $\mathbb{Z}_{2(n-1)}$. The kernel of the associated projection $\xi\colon\ B_n(\mathbb{S}^2) \to \mathbb{Z}_{2(n-1)}$ (which is defined by $\xi(\sigma_i) = \overline{1}$ for all $1 \leq i \leq n-1$) is the commutator subgroup $\Gamma_2\left(B_n(\mathbb{S}^2)\right)$. If $w \in B_n(\mathbb{S}^2)$ then $\xi(w)$ is the exponent sum (relative to the σ_i) of w modulo $2(n-1)$.

The torsion elements of the braid groups of $\mathbb{S}^2$ and $\mathbb{R}P^2$ were classified by Murasugi [M]: if $M = \mathbb{S}^2$ and $n \geq 3$, they are all conjugates of powers of the three elements $\alpha_0 = \sigma_1 \cdots \sigma_{n-2}\sigma_{n-1}$ (which is of order $2n$), $\alpha_1 = \sigma_1 \cdots \sigma_{n-2}\sigma_{n-1}^2$ (of order $2(n-1)$) and $\alpha_2 = \sigma_1 \cdots \sigma_{n-3}\sigma_{n-2}^2$ (of order $2(n-2)$) which are respectively n^{th}, $(n-1)^{\text{th}}$ and $(n-2)^{\text{th}}$ roots of T_n, where T_n is the so-called 'full twist' of $B_n(\mathbb{S}^2)$, defined by $T_n = (\sigma_1 \cdots \sigma_{n-1})^n$. If $n \geq 3$, T_n is the unique element of $B_n(\mathbb{S}^2)$ of order 2 and generates the centre of $B_n(\mathbb{S}^2)$. In [GG2], we showed that $B_n(\mathbb{S}^2)$ is generated by α_0 and α_1.

For $n \geq 4$, $B_n(\mathbb{S}^2)$ is infinite. It is an interesting question as to which finite groups are realised as subgroups of $B_n(\mathbb{S}^2)$ (apart of course from the cyclic groups $\langle \alpha_i \rangle$). In [GG2], we proved that $B_n(\mathbb{S}^2)$ contains an isomorphic copy of the finite group $B_3(\mathbb{S}^2)$ of order 12 if and only if $n \not\equiv 1 \bmod 3$. The quaternion group $\mathcal{Q}_8$ of order 8 appears in the study of braid groups of non-orientable surfaces, being isomorphic to the 2-string pure braid group $P_2(\mathbb{R}P^2)$. Further, since the projection $F_3(\mathbb{R}P^2) \to F_2(\mathbb{R}P^2)$ of configuration spaces of $\mathbb{R}P^2$ onto the first two coordinates admits a section [VB], it follows using the Fadell-Neuwirth short exact sequence that $P_3(\mathbb{R}P^2)$ is a semi-direct product of a free group of rank 2 by $\mathcal{Q}_8$ [GG1].

While studying the lower central and derived series of the sphere braid groups, we showed that $\Gamma_2\left(B_4(\mathbb{S}^2)\right)$ is isomorphic to a semi-direct product of $\mathcal{Q}_8$ by a free group of rank 2 [GG3]. After having proved this result, we noticed that the question of the realisation of $\mathcal{Q}_8$ as a subgroup of $B_n(\mathbb{S}^2)$ was explicitly posed by R. Brown in connection with the fact that the fundamental group of SO(3) is isomorphic to $\mathbb{Z}_2$ [ATD]. In this paper, we give a complete answer to this question:

Theorem. *Let $n \in \mathbb{N}$, $n \geq 3$.*

(a) If n is a multiple of 4 then $\Gamma_2\left(B_n(\mathbb{S}^2)\right)$ contains a subgroup isomorphic to $\mathcal{Q}_8$.
(b) If n is an odd multiple of 2 then $B_n(\mathbb{S}^2)$ contains a subgroup isomorphic to $\mathcal{Q}_8$.
(c) If n is odd then $B_n(\mathbb{S}^2)$ contains no subgroup isomorphic to $\mathcal{Q}_8$.

Proof. We first suppose that n is even, so that $n = 2m$ with $m \in \mathbb{N}$. Let H be the subgroup of $B_{2m}(\mathbb{S}^2)$ generated by x and y, where

$$
\begin{aligned}
x =&(\sigma_1 \cdots \sigma_{2m-1})(\sigma_1 \cdots \sigma_{2m-2}) \cdots (\sigma_1\sigma_2)\sigma_1,\\
y =&(\sigma_1 \cdots \sigma_{m-1})(\sigma_1 \cdots \sigma_{m-2}) \cdots (\sigma_1\sigma_2)\sigma_1 \cdot \sigma_{2m-1}^{-1}(\sigma_{2m-2}^{-1}\sigma_{2m-1}^{-1}) \cdots\\
&\cdots (\sigma_{m+2}^{-1} \cdots \sigma_{2m-1}^{-1})(\sigma_{m+1}^{-1} \cdots \sigma_{2m-1}^{-1}).
\end{aligned}
$$

Geometrically, x may be interpreted as the 'half twist' or Garside element of B_{2m} [Bi]. Further, y may be considered as the commuting product of the positive half twist of the first m strings with the negative half twist of the last m strings. Then $x^2 = T_{2m}$ and $y^2 = (\sigma_1 \ldots \sigma_{m-1})^m(\sigma_{m+1}^{-1} \ldots \sigma_{2m-1}^{-1})^m = T_{2m}$ in $B_{2m}(\mathbb{S}^2)$ (cf. [FVB, GVB]). It is well known that $x\sigma_i x^{-1} = \sigma_{2m-i}$ in B_{2m} [Bi], and thus in $B_{2m}(\mathbb{S}^2)$, from which we obtain $xyx^{-1} = y^{-1}$. Hence H is isomorphic to a quotient of $\mathcal{Q}_8$. But x is of order 4, and the induced permutation of y on the symmetric group S_{2m} is different from that of the elements of $\langle x \rangle$. It follows that H contains the five distinct elements of $\langle x \rangle \cup \{y\}$, and so $H \cong \mathcal{Q}_8$. If m is even then $x, y \in \operatorname{Ker}(\xi)$, and thus $H \subseteq \Gamma_2(B_n(\mathbb{S}^2))$. This proves parts (a) and (b).

To prove part (c), suppose that n is odd, and suppose that $x, y \in B_n(\mathbb{S}^2)$ generate a subgroup H isomorphic to $\mathcal{Q}_8$, so $x^2 = y^2$ and $xyx^{-1} = y^{-1}$. In particular, x and y are of order 4, and thus are conjugates of $\alpha_1^{\pm(n-1)/2}$ by Murasugi's classification. By considering a conjugate of H if necessary, we may suppose that $x = \alpha_1^{\varepsilon_1(n-1)/2}$ and $y = w\alpha_1^{\varepsilon_2(n-1)/2}w^{-1}$, where $w \in B_n(\mathbb{S}^2)$ and $\varepsilon_1, \varepsilon_2 \in \{1, -1\}$. Replacing x by x^{-1} if necessary, we may suppose further that $\varepsilon_1 = -\varepsilon_2$. Thus $xy = \left[\alpha_1^{\varepsilon_1(n-1)/2}, w\right]$, and is of exponent sum zero. On the other hand, xy is an element of H of order 4, and so is conjugate to $\alpha_1^{\pm(n-1)/2}$ by Murasugi's classification. But $\xi\left(\alpha_1^{\pm(n-1)/2}\right) = \pm\frac{n(n-1)}{2}$, which is non zero modulo $2(n-1)$. This yields a contradiction, and proves part (c). $\square$

Remark. Let $n \geq 3$. Using techniques similar to those of the proof of the Theorem, one may show that the subgroup of $B_n(\mathbb{S}^2)$ generated by $\sigma_1 \cdots \sigma_{n-1}$ and the half twist x is isomorphic to the dicyclic group of order $4n$. In particular, if n is a power of two then $B_n(\mathbb{S}^2)$ contains a subgroup isomorphic to the generalised quaternion group of order $4n$. Further investigation into the finite subgroups of $B_n(\mathbb{S}^2)$ and $B_n(\mathbb{R}P^2)$ will appear elsewhere.

References

[ATD] Algebraic topology discussion list, January 2004, `http://www.lehigh.edu/~dmd1/pz119.txt`.

[A1] E. Artin, Theorie der Zöpfe, *Abh. Math. Sem. Univ. Hamburg* **4** (1925), 47–72.

[A2] E. Artin, Theory of braids, *Ann. Math.* **48** (1947), 101–126.

[Bi] J. S. Birman, Braids, links and mapping class groups, *Ann. Math. Stud.* **82**, Princeton University Press, 1974.

[FVB] E. Fadell and J. Van Buskirk, The braid groups of $\mathbb{E}^2$ and $\mathbb{S}^2$, *Duke Math. Journal* **29** (1962), 243–257.

[FoN] R. H. Fox and L. Neuwirth, The braid groups, *Math. Scandinavica* **10** (1962), 119–126.

[GVB] R. Gillette and J. Van Buskirk, The word problem and consequences for the braid groups and mapping class groups of the 2-sphere, *Trans. Amer. Math. Soc.* **131** (1968), 277–296.

[GG1] D. L. Gonçalves and J. Guaschi, The braid groups of the projective plane, *Algebraic and Geometric Topology* **4** (2004), 757–780.

[GG2] D. L. Gonçalves and J. Guaschi, The braid group $B_{n,m}(\mathbb{S}^2)$ and the generalised Fadell-Neuwirth short exact sequence, *J. Knot Theory and its Ramifications* **14** (2005), 375–403.

[GG3] D. L. Gonçalves and J. Guaschi, The lower central and derived series of the braid groups $B_n(\mathbb{S}^2)$ and $B_m(\mathbb{S}^2 \setminus \{x_1, \ldots, x_n\})$, preprint, March 2006.

[M] K. Murasugi, Seifert fibre spaces and braid groups, *Proc. London Math. Soc.* **44** (1982), 71–84.

[VB] J. Van Buskirk, Braid groups of compact 2-manifolds with elements of finite order, *Trans. Amer. Math. Soc.* **122** (1966), 81–97.